UNSTEADY FLOW OF A VISCOUS INCOMPRESSIBLE LIQUID BETWEEN TWO CYLINDERS

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A study has been made of the unsteady motion of a viscous incompressible liquid between two cylinders, the outer one being of arbitrary, slightly non-circular shape, while the inner one is circular and rotates about the origin. Results have been obtained which account for certain phenomena occurring in high-speed rotary hydraulic bearings.

Let us examine a flow of viscous incompressible liquid between two cylinders, caused by the motion of the inner cylinder. Let the shape of the outer cylinder differ slightly from the circular, i.e., the equation of this cylinder can be written in polar co-ordinates as:

$$r = r_2 + r_2 \varepsilon_1 f(\vartheta). \tag{1}$$

In plan the shape of the inner cylinder is a circle of radius r_1. Its center describes a circle of radius $r_1\varepsilon_2$. Let the plane of the complex variable z represent a transverse section perpendicular to the cylinders.

In the plane z the region occupied by the liquid is bounded by an arbitrary closed curve having two finite derivatives and differing slightly from a circle centered at $z_1 = -r_1\varepsilon_2\exp[i\,\omega\alpha(t)]$.

Assuming that ε_1 and ε_2 are small quantities of the same order, we can restrict ourselves to only the first order, and the equation of the inner contour becomes

$$r = r_1 + r_1 \varepsilon_2 \cos(\vartheta - \omega\alpha(t)). \tag{2}$$

The region bounded by curves (1) and (2) differs slightly from a circular ring. The mapping of such regions on a circular ring has been examined in [1]. Using the properties of elliptic functions [2], the mapping function can be written thus:

$$\begin{aligned}\zeta = (z/r_1)\exp[-i\,\omega\alpha(t)]\,[1 + a_0\delta_0 + \sum_{n=1}^{\infty}\delta_n (z/r_1)^n (a_n - ib_n) - \\ - \sum_{n=1}^{\infty}\delta_n (r_1/z)^n (a_n + ib_n)] + \delta - \varepsilon_2 - \delta\,(z/r_1)^2 \exp[-2i\,\omega\alpha(t)],\end{aligned} \tag{3}$$

where small quantities of the first order* only have been retained and

$$a_0 = 1/2\pi \int_0^{2\pi} f(k)\,dk; \quad a_n = 1/\pi \int_0^{2\pi} f(k)\cos nk\,dk; \quad b_n = 1/\pi \int_0^{2\pi} f(k)\sin nk\,dk;$$

$$\delta = -\frac{\varepsilon_2 r_1^2}{r_2^2 - r_1^2}; \quad \delta_0 = -\varepsilon_1\left(\frac{r_1}{r_2}\right)^2; \quad \delta_n = -\varepsilon_1 \frac{(r_1 r_2)^n}{r_2^{2n} - r_1^{2n}}.$$

In the ζ plane the region bounded by (1) and (2) becomes a ring bounded by circles $\rho = 1$ and $\rho = \beta(\beta > 1)$, where

$$\frac{1}{\beta} = \frac{r_1}{r_2}\left[1 - \frac{r_2^2 - r_1^2}{r_1^2}\varepsilon_1 a_0\right].$$

Before writing the equation of motion in the ζ plane, we note that if u and v are the velocity components in the directions ρ and φ ($\zeta = \rho\exp(i\varphi)$) respectively, the continuity equation may be written

$$\frac{\partial \rho J^{1/2} u}{\partial \rho} + \frac{\partial J^{1/2} v}{\partial \varphi} = 0, \tag{4}$$

* Because of the awkwardness of the second approximation, the solution is given only for the first approximation.

where the Jacobian J and the dimensionless Jacobian J_x are given by the relations

$$J = r_1^2 J_x = \left|\frac{dz}{d\zeta}\right|^2 = r_1^2\Big\{ 1 + 4\delta\rho\cos\varphi + 2a_0\delta_0 +$$

$$+ 2\sum_{n=1}^{\infty}\delta_n[(n+1)\rho^n + (n-1)\rho^{-n}][a_n\cos n(\varphi+\omega\alpha) +$$

$$+ b_n\sin n(\varphi+\omega\alpha)]\Big\}. \tag{5}$$

From the continuity equation we take the dimensionless current function ψ

$$u = r_1^2\omega\rho^{-1}J^{-1/2}\frac{\partial\psi}{\partial\varphi};\quad v = -r_1^2\omega J^{-1/2}\frac{\partial\psi}{\partial\rho}. \tag{6}$$

Now the generalized Helmholtz equation (3) may easily be written in the ζ plane

$$\mathrm{Re}\frac{\partial}{\partial t}(\Delta\psi/J_x) + \frac{\mathrm{Re}}{J_x\rho}\,\frac{D(\Delta\psi/J_x,\ \psi)}{D(\rho,\ \varphi)} = \frac{1}{J_x}\Delta(\Delta\psi/J_x). \tag{7}$$

In formulating boundary conditions for equation (7), we assume, without loss of generality, that $\alpha(0) = 0$. Then, if at time t = 0 points of the inner circle occupy the position $z(0) = r_1\varepsilon_2 + r_1\exp(i\vartheta)$, at time t these points will occupy the position $z(t) = r_1\varepsilon_2 \times \exp[i\omega\alpha(t)] + r_1\exp(i\vartheta)$ and will have normal and tangential velocities $\alpha'\omega r_1\varepsilon_2\sin(\vartheta-\omega\alpha)$ and $\omega_1 r_1 + \alpha'\omega r_1\varepsilon_2\cos(\vartheta-\omega\alpha)$, respectively.

In a particular case ω_1 may be zero. When $z = r_1\exp(i\vartheta) + r_1\varepsilon_2\exp[i\omega\alpha(t)]$

$$\zeta = \exp(i\varphi) = \exp[i(\vartheta-\omega\alpha)]\Big\{1 + \delta[1-\exp(2i(\vartheta-\omega\alpha))] + a_0\delta_0 +$$

$$+ \sum_{n=1}^{\infty}\delta_n[(a_n - ib_n)\exp(in\vartheta) - (a_n+ib_n)\exp(-in\vartheta)]\Big\},$$

from which we easily find

$$\cos(\vartheta-\omega\alpha) = \cos\varphi + a_0\delta_0\cos\varphi +$$

$$+ 2\sum_{n=1}^{\infty}\delta_n\sin\varphi[a_n\sin n(\varphi+\omega\alpha) - b_n\cos n(\varphi+\omega\alpha)] - \delta(1-\cos 2\varphi);$$

$$\sin(\vartheta-\omega\alpha) = \sin\varphi + a_0\delta_0\sin\varphi +$$

$$+ 2\sum_{n=1}^{\infty}\delta_n\cos\varphi[a_n\sin n(\varphi+\omega\alpha) - b_n\cos n(\varphi+\omega\alpha)] + \delta\sin 2\varphi, \tag{8}$$

where small quantities of the first order only are retained.

Then the boundary conditions for ψ may be written

$$\rho = 1,\quad \frac{\partial\psi}{\partial\varphi} = -(\beta^2-1)\alpha'\delta\sin\varphi);$$

$$\frac{\partial\psi}{\partial\rho} = -\Omega\Big\{1 + a_0\delta_0 + 2\delta\cos\varphi + 2\sum_{n=1}^{\infty}\delta_n n[a_n\cos n(\varphi+\omega\alpha) +$$

$$+ b_n\sin n(\varphi+\omega\alpha)]\Big\} + \alpha'(\beta^2-1)\delta\cos\varphi; \tag{9}$$

$$\rho = \beta,\quad \frac{\partial\psi}{\partial\rho} = \frac{\partial\psi}{\partial\varphi} = 0,$$

where $\Omega = \omega_1/\omega$.

Now, for the full solution we require the obvious condition: that the hydraulic pressure be a periodic function of the variables φ or ϑ. Equation (7) is unsteady, and it is therefore necessary to add an initial condition: $\psi(0, \rho\varphi)$ is given when $t = 0$. Equation (7) is a very complex nonlinear equation. This paper deals only with the case when ε_1 and ε_2 are small quantities and ε_i^2 ($i = 1, 2$) may be neglected in comparison with ε_i; the same applies to δ and δ_n by definition. The solution of (7) takes the form of a power series in the small parameters δ and δ_n, and terms of higher order than a chosen term are neglected. The solution of (7) is sought in the form

$$\psi = \psi_0 + \delta\,\psi_1 + \sum_{n=1}^{\infty} \delta_n \psi_{1n}, \tag{10}$$

where $\psi_0 = \dfrac{\Omega}{\beta^2 - 1}\left(\dfrac{1}{2}\rho^2 - \beta^2 \ln \rho\right)$ + const is the solution of (7) for the case when δ and δ_n ($n = 0, 1, 2, \ldots$) are zero, with corresponding boundary conditions. By substituting (10) in (7) and (9) respectively we obtain in the usual way the following series of equations together with their boundary conditions:

$$\begin{gathered}
\Delta\Delta\psi_1 + \frac{\operatorname{Re}\Omega}{\beta^2 - 1}\left(1 - \frac{\rho^2}{\beta^2}\right)\frac{\partial \Delta\psi_1}{\partial \varphi} - \operatorname{Re}\frac{\partial \Delta\psi_1}{\partial t} - \\
- 8\frac{\operatorname{Re}\Omega^2}{(\beta^2 - 1)^2}\left(1 - \frac{\beta^2}{\rho^2}\right)\rho \sin\varphi = 0; \\
\rho = 1, \quad \frac{\partial \psi_1}{\partial \varphi} = -\alpha'(\beta^2 - 1)\sin\varphi, \quad \frac{\partial \psi_1}{\partial \rho} = [\alpha'(\beta^2 - 1) - 2\Omega]\cos\varphi; \\
\rho = \beta, \quad \frac{\partial \psi_1}{\partial \varphi} = \frac{\partial \psi_1}{\partial \rho} = 0; \\
\Delta\Delta\psi_{10} + \frac{\operatorname{Re}\Omega}{\beta^2 - 1}\left(1 - \frac{\beta^2}{\rho^2}\right)\frac{\partial \Delta\psi_{10}}{\partial \varphi} - \operatorname{Re}\frac{\partial \Delta\psi_{10}}{\partial t} = 0; \\
\rho = 1, \quad \frac{\partial \psi_{10}}{\partial \varphi} = 0, \quad \frac{\partial \psi_{10}}{\partial \rho} = -\Omega a_0; \\
\rho = \beta; \quad \frac{\partial \psi_{10}}{\partial \varphi} = \frac{\partial \psi_{10}}{\partial \rho} = 0; \\
\Delta\Delta\psi_{1n} + \frac{\operatorname{Re}\Omega}{\beta^2 - 1}\left(1 - \frac{\beta^2}{\rho^2}\right)\frac{\partial \Delta\psi_{1n}}{\partial \varphi} - \operatorname{Re}\frac{\partial \Delta\psi_{1n}}{\partial t} - \\
- \frac{4\operatorname{Re}\Omega}{\beta^2 - 1} n\left[\frac{\Omega}{\beta^2 - 1}\left(1 - \frac{\beta^2}{\rho^2}\right) - \alpha'\right] \times \\
\times [(n+1)\rho^n + (n-1)\rho^{-n}][a_n \sin n(\varphi + \omega\alpha) - b_n \cos n(\varphi + \omega\alpha)] = 0; \\
\rho = 1, \quad \frac{\partial \psi_{1n}}{\partial \varphi} = 0, \\
\frac{\partial \psi_{1n}}{\partial \rho} = -2\Omega n[a_n \cos n(\varphi + \omega\alpha) + b_n \sin n(\varphi + \omega\alpha)]; \\
\rho = \beta, \quad \frac{\partial \psi_{1n}}{\partial \varphi} = \frac{\partial \psi_{1n}}{\partial \rho} = 0 \quad (n = 1, 2, 3 \ldots).
\end{gathered} \tag{11}$$

Case α'= const. We shall confine ourselves to the particular case mentioned above, as being of most interest from the applied point of view. Let us assume that the inner cylinder rotates about the origin of coordinates with a constant angular velocity ω, i.e., that $\alpha(t) = t$ and $\alpha' = 1$. In this case some of the equations and boundary conditions of (11) are partially simplified:

$$\begin{gathered}
\Delta\Delta\psi_1 + \frac{\operatorname{Re}\Omega}{\beta^2 - 1}\left(1 - \frac{\beta^2}{\rho^2}\right)\frac{\partial \Delta\psi_1}{\partial \varphi} - 8\frac{\operatorname{Re}\Omega^2}{(\beta^2 - 1)^2}\left(1 - \frac{\beta^2}{\rho^2}\right)\rho \sin\varphi = 0; \\
\rho = 1, \quad \frac{\partial \psi_1}{\partial \varphi} = -(\beta^2 - 1)\sin\varphi, \quad \frac{\partial \varphi_1}{\partial \rho} = [(\beta^2 - 1) - 2\Omega]\cos\varphi;
\end{gathered} \tag{12}$$

$$\rho=\beta,\quad \frac{\partial\psi_1}{\partial\varphi}=\frac{\partial\psi_1}{\partial\rho}=0;\quad \Delta\Delta\psi_{10}+\frac{\operatorname{Re}\Omega}{\beta^2-1}\left(1-\frac{\beta^2}{\rho^2}\right)\frac{\partial\Delta\psi_{10}}{\partial\varphi}=0;$$

$$\rho=1,\quad \frac{\partial\psi_{10}}{\partial\varphi}=0,\quad \frac{\partial\psi_{10}}{\partial\rho}=-a_0\Omega; \tag{13}$$

$$\rho=\beta,\quad \frac{\partial\psi_{10}}{\partial\varphi}=\frac{\partial\psi_{10}}{\partial\rho}=0;$$

$$\Delta\Delta\psi_{1n}+\frac{\operatorname{Re}\Omega}{\beta^2-1}\left(1-\frac{\beta^2}{\rho^2}\right)\frac{\partial\Delta\psi_{1n}}{\partial\varphi}-\operatorname{Re}\frac{\partial\Delta\psi_{1n}}{\partial t}-$$

$$-4\frac{\operatorname{Re}\Omega}{\beta^2-1}n\left[\frac{\Omega}{\beta^2-1}\left(1-\frac{\beta^2}{\rho^2}\right)-1\right][(n+1)\rho^n+(n-1)\rho^{-n}]\times$$

$$\times[a_n\sin n(\varphi+\omega t)-b_n\cos n(\varphi+\omega t)]=0; \tag{14}$$

$$\rho=1,\quad \frac{\partial\varphi_{1n}}{\partial\varphi}=0,\quad \frac{\partial\psi_{1n}}{\partial\rho}=-2\Omega n\{a_n\cos n(\varphi+\omega t)+b_n\sin n(\varphi+\omega t)\};$$

$$\rho=\beta,\quad \frac{\partial\psi_{1n}}{\partial\varphi}=\frac{\partial\psi_{1n}}{\partial\rho}=0.$$

Because of their structure a solution of (12), (13) and (14) should be sought in the form:

$$\psi_1=\frac{\Omega}{\beta^2-1}\rho^3\cos\varphi+\operatorname{Real}\{f(\rho)\exp(-i\varphi)\}+h_{01}(\rho); \tag{15}$$

$$\psi_{10}=\frac{a_0\Omega}{\beta^2-1}\left[\frac{1}{2}\rho^2-\beta^2\ln\rho\right]+h_{02}(\rho); \tag{16}$$

$$\psi_{1n}=\frac{\Omega}{\beta^2-1}\rho^2[\rho^n-\rho^{-n}l_n]\{a_n\cos n(\varphi+\omega t)+b_n\sin n(\varphi+\omega t)\}+$$

$$+\operatorname{Real}\{f_n(\rho)(a_n+ib_n)\exp[-i(n\varphi+\omega t)]\}+h_{1n}(\rho), \tag{17}$$

where $f(\rho)$ is defined as in [4] and [5], and $f_n(\rho)$ may be determined by substituting (17) in (14) to obtain the following equation and boundary conditions:

$$g_n''+\frac{1}{\rho}g_n'-\left[in\left(1+\frac{\operatorname{Re}\Omega}{\beta^2-1}\right)+\right.$$

$$\left.+n^2\left(1+\frac{i\beta^2\operatorname{Re}\Omega}{n(\beta^2-1)}\right)\rho^{-2}\right]g_n=0; \tag{18}$$

$$f_n''+\frac{1}{\rho}f_n'-\frac{n^2}{\rho^2}f_n=g_n;$$

$$f_n(1)=-\frac{\Omega}{\beta^2-1}(1-l_n);$$

$$f_n'(1)=-2\Omega n-[2(1-l_n)+n(1+l_n)]\frac{\Omega}{\beta^2-1}; \tag{19}$$

$$f_n(\beta)=-\frac{\Omega}{\beta^2-1}(\beta^{n+2}-\beta^{-n+2}l_n);$$

$$f_n'(\beta)=-\frac{\Omega}{\beta^2-1}[(n+2)\beta^{n+1}+l_n(n-2)\beta^{-n+1}],$$

where $l_1=0$ and $l_n=1$ when $n>1$.

The general solution of (18) may be written in the form

$$f_n(\rho)=A_n\rho^n+B_n\rho^{-n}+C_nI_{1n}(\rho)+D_nI_{2n}(\rho), \tag{20}$$

where $I_{1n}(\rho)$ and $I_{2n}(\rho)$ are expressed in parallel manner in terms of Bessel functions (5). To determine A_n, B_n, C_n, and D_n we substitute (20) in (19) to obtain an algebraic system of linear equations. Solving this system, we obtain:

$$A_n = F_1(\beta, \mathrm{Re}, n), \quad B_n = F_2(\beta, \mathrm{Re}, n),$$
$$C_n = F_3(\beta, \mathrm{Re}, n) \text{ и } D_n = F_4(\beta, \mathrm{Re}, n).$$

Because of the awkwardness of functions $F_i(\beta, \mathrm{Re}, n)$ they have not been written out.

By virtue of the boundary conditions for the respective equations, and the obvious condition (hydraulic pressure is a periodic function of φ), functions $h_{ij}(\rho)$ are equal to arbitrary constants. Thus the solution is determined correct to arbitrary constants, which have no influence on the velocity field.

Knowing the function Ψ, we can determine the force to which the inner cylinder is subjected in its motion.

Determination of force. We introduce the following fixed coordinate system: the x axis is directed along the line joining the origin of coordinates to the center of the inner cylinder, and the y axis is perpendicular to it.

The expressions for the projection on the moving axes Ox and Oy of the principal vector of the forces acting on the inner cylinder are:

$$p_x = \int_0^{2\pi} [P_{\rho\rho} \cos(x, \rho) + P_{\rho\varphi} \cos(x, \varphi)]_{\rho=1} J^{1/2}(1, \varphi)\, d\varphi,$$
$$p_y = \int_0^{2\pi} [P_{\rho\rho} \cos(y, \rho) + P_{\rho\varphi} \cos(y, \varphi)]_{\rho=1} J^{1/2}(1, \varphi)\, d\varphi. \tag{21}$$

Carrying out the simple, but laborious calculations, we obtain expressions for the forces:

$$p_x = \varepsilon_2 p_{1x}(\mathrm{Re}, \beta) + \mathrm{Reel}\,[\varepsilon_1 p_{2x}(\mathrm{Re}, \beta)(a_1 + ib_1) \exp(i\omega t)],$$
$$p_y = \varepsilon_2 p_{1y}(\mathrm{Re}, \beta) + \mathrm{Reel}\,[\varepsilon_1 p_{2y}(\mathrm{Re}, \beta)(a_1 + ib_1) \exp(i\omega t)]. \tag{22}$$

Because the expressions for $p_{ix}(\mathrm{Re}, \beta)$ and $p_{iy}(\mathrm{Re}, \beta)$ are very unwieldy, they have not been given in this paper.

Only the first Fourier coefficients, boundary deviation functions linked with the orthogonality of sines and cosines, enter into formula (22). The real part of $(A + iB) = A$, where A and B are real.

It follows from (22) that in its motion the inner cylinder experiences a force directly proportional to the deviation of the inner cylinder from the coordinate origin and to the deviation of the outer cylinder from the circular. Only two Fourier coefficients enter into the formula for the force. If ε_1 is put equal to zero, we have the case of a cylinder rotating within a cylinder. It follows from (22) that, in addition to the force acting perpendicular to the line of centers, there is a force directed along the line of centers. This result coincides with the conclusions of others [6] who have studied the influence of inertia forces on the inner cylinder. By means of (22) we can evaluate the contribution to these forces arising from deviation of the shape of the outer cylinder from the circular.

Some limiting cases. Let us examine some limiting cases of the results obtained. In the first place, let us take the case Re = 0. Then (7) becomes

$$\Delta(\Delta\psi / J_*) = 0 \tag{23}$$

with boundary conditions (9). Let us assume that $\varepsilon_1 = 0$, then $\alpha' = \text{const}$, and ε is sufficiently small. Putting the solution (23) in the form of a series, and again confining ourselves to the first terms of the expansion, for the force exerted by the liquid on the inner cylinder we arrive at conclusions which coincide with the Chaplygin-Joukowski theory for corresponding assumptions [7].

Let us consider another limiting case. Let Re become arbitrarily large. Writing down the asymptotic solution of (7) for arbitrarily large Re, and using (22), we obtain the following formulas for the principal vector of the forces in dimensionless form

$$p_y = 0, \quad p_x = \pi \left[4\Omega^2 - (\beta^2 - 3)\Omega\alpha' + \frac{3\Omega^2}{\beta^2 - 1} + \frac{2\Omega^2}{(\beta^2 - 1)^2}\right]\delta. \tag{24}$$

In contrast to the Chaplygin-Joukowski theory, in this limiting case, we find no force perpendicular to the line of centers.

NOTATION

$1/\omega$ – characteristic time associated with motion; ε_1 and ε_2 – dimensionless constants; $\alpha(t)$ – arbitrary functions of time; $Re = r_1^2\omega/\nu$ – Reynolds number; ν – kinematic vicsocity; ω_1 – angular velocity of rotation of inner cylinder about its center.

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